THE TRISECTION OF THE ANGLE BY ABÛ SAHL WAYJAN IBN RUSTAM AL KÛHÎ (fl. 970-988)*

In an Istanbul manuscript which is more than nine hundred years old (Ayasofya Museum Library, No. 4832 [1]) there is a short tract giving a method found by Al Kûhî for the trisection of the angle. It is on page 147b of this sizeable volume which is a collection of a large number of articles and short treatises. The present text occupies only part of a single page.

For the trisection of the angle Al Kûhî utilizes here a problem of the construction, in a given plane, of a hyperbola in which the angle between a diameter and the corresponding ordinates, called the angle of arrangement by Al Kûhî, equals a given angle and whose parameter corresponding to the particular diameter and the diameter itself have given values [2]. Al Kûhî's method of trisection calls for the construction of such a hyperbola whose angle of arrangement is equal to the angle to be trisected and in which the parameter and the diameter are equal to each other.

Let AB be the hyperbola and ADV its angle of arrangement, the ordinate of the point A being AD and the corresponding diameter CB. Furthermore, let A be such a point that AB is equal in value to the parameter. $AB = BC$, therefore, and the triangle ABC is isoceles.

As by a well-known proposition $\frac{AD^2}{DB.\,DC} = \frac{\text{parameter}}{CB}$ and as in our hyperbola parameter $= CB$, we have $AD^2 = DB\,.\,DC$ and $\frac{DB}{AD} = \frac{AD}{DC}$. The triangles ABD and ACD which have one angle in common are therefore similar and the angles

* The contents of the manuscript studied in the present article formed the subject of my paper presented at the Tenth International Congress of the History of Science held between August 26, 1962 and September 2, 1962, in Ithaca, New York, and Philadelphia, Pennsylvania.

[1] For more detail on this manuscript, see, Aydın Sayılı, *Thâbit ibn Qurra's Generalization of the Pythagorean Theorem*, *Isis*, vol. 51, 1960, p. 35, note 1.

[2] Apollonios, *Conic Sections*, English translation by T.L. Heath, propositio n25.

ABD and CAD are equal. But $< \text{ABD} = 2 < \text{C}$. Therefore $< \text{CAD} = 2\text{C}$. Now, $< \text{ADV} = < \text{CAD} + \text{C} = 2\text{C} + \text{C} = 3\text{C}$, and $< \text{C} = \frac{1}{3} < \text{ADV}$. The angle ADV has therefore been trisected.

In the text of the manuscript about half the dots of letters are missing. I have added these as well as the *mad* and *hamza* signs in the printed text. The manuscript figure contains also a separate figure for the angle to be trisected (angle ه in the text and H in translations). This angle appears as approximately equal to the supplement of the angle ADV.

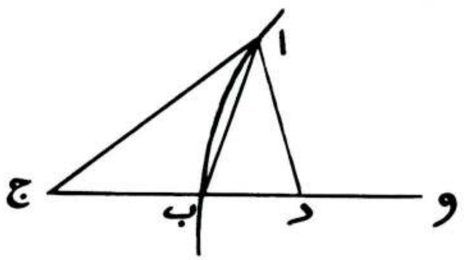
The part of the title of our Arabic text below appearing in brackets is added on the basis of the title of another short article by Al Kûhî which immediately precedes the present one in the manuscript. The part of the concluding note shown in parentheses, at the end of our text, likewise refers to that preceding article.

ARAPÇA METİN
(Arabic Text)

فى قسمة الزاوية بثلثة اقسام

متساوية له [لابى سهل الكوهى]

نضع الزاوية زاوية «ه» ونعمل قطعاً زائداً عليه «ا ب» وليكن ضلعاه القائم والمائل متساويين وليكن «ا ب» مثل ضلعه القائم و«ب ج» هو ضلعه المائل ولتكن زاوية ترتيبه مثل زاوية «ه» وذلك كماتبين فى آخر مقالة [1] «ا» من كتاب المخروطات عند عمله القطع الزائد فلان نسبة ضرب «ج د» فى «د ب» الى مربع «ا د» كنسبة الضلع القائم الى المائل كماتبين فى شكل «ك» من مقالة «ا» من المخروطات يكون ضرب «ج د» فى «د ب» مثل مربع «ا د» فزاوية «ا د ب» اذن مثل زاوية «ج ا د» وزاوية «د ب ا» مثلا زاوية «ا ج د» لان «ا ب» مثل «ب ج» فزاوية «ج ا د» اذن مثلا زاوية «ا ج د» فزاويتا «ج ا د» «ا ج د» مجموعتان ثلثة امثال زاوية «ا ج د» لكن زاوية «ا د و» الخارجة مثل زاويتى «د ا ج» «ا ج د» الداخلتين مجموعتين من المثلث فزاوية «ا د و» ثلثة امثال زاوية «ج» فزاوية «ج» اذن ثلث زاوية «ا د و» التى هى زاوية ترتيب القطع وقد كنا فرضنا ان زاوية الترتيب مثل زاوية «ه» فيجب اذن ان تكون زاوية «ج» ثلث زاوية «ه» المفروضة فقد وجدنا ثلث زاوية «ه» وذلك ما اردنا ان نبين.



تم (استخراج الخطين بين الخطين على نسبة و) قسمة الزاوية بثلثة اقسام متساوية لابى سهل الكوهى والحمد لله رب العالمين وصلواته على نبيه محمد وآله الطاهرين.

1 ــ المقالة.

TÜRKÇE TERCÜME

Açının Üç Eşit Kısma Bölünmesi
Ebû Sehl el Kûhî Tarafından

Bölünecek açı H açısı olsun. Bu açı üzerine bir AB hiperbolü çizelim. Bu hiperbolün parametresi ile köşegeni birbirlerine, AB doğru parçası da parametreye eşit olsun. BC bu hiperbolün köşegeni, H (yani ADV'ye eşit olan ve üçe bölünmesi istenen açı)de tertip açısı olsun. Bu hiperbolün böylece çizimi (Apollonios'un) Koni Kesitleri kitabının birinci kısmının sonunda [1] hiperbol çizimi üzerinde verdiği tafsilâta göre yapılır.

Bu hiperbolde, aynı kitabın birinci kısmının yirminci propozisyonunda belirtildiği üzere, [2] CD'nin DB ile çarpımının AD^2'ye oranı parametrenin köşegene oranı gibi olduğundan, CD'nin DB ile çarpımı AD'nin karesine eşit olur. Böyle olunca da ABD açısı CAD açısına, DBA açısı da ACD açısının iki katına eşit olur. Çünkü AB doğru parçası BC'ye eşittir. Şu halde CAD açısı ACD açısının iki katına, ve CAD ile ACD açıları toplamı ACD açısının üç katına eşittir. Öte yandan, üçgenin ADV dış açısı DAC ve ACD iç açıları toplamına eşittir. Demek ki ADV açısı C açısının üç katı, C açısı da hiperbolümüzün tertip açısı olan ADV açısının üçte biridir. Tertip açısının H açısına eşit olduğunu kabul etmiştik. Şu halde C'nin H açısının üçte biri olması gerekmektedir. Böylece H açısının üçte birini elde etmiş bulunuyoruz. Bu da açıklamak istediğimiz husustur.

Ebû Sehl el Kûhî'nin... açıyı üç eşit kısma bölme üzerine risalesi sona erdi. Hamd dünyaların tanrısı Allaha'dır. Onun salâtı peygamberi Muhammed'e ve temiz ailesi fertlerine olsun.

[1] Apollonios, *Conic Sections*, T.L. Heath, s. 44-47.
[2] Bk., Apollonios, aynı kitap, s. 10, 19-20.

ENGLISH TRANSLATION

Concerning the Division of the Angle into Three Equal Parts by Him (Abû Sahl al Kûhî)

We take the angle as the angle H (i. e., the angle to be trisected to which is set equal ADV) and construct upon it a hyperbola AB. Let the latus rectum (or parameter) and the diameter of this hyperbola be equal to each other, and let the line AB also be equal to the parameter, BC being the diameter. Let the angle between the ordinate and the diameter be equal to the angle H. This is done as indicated at the end of book l of the Conic Sections,[1] i.e., in the part dealing with the construction of the hyperbola.

Now, since the ratio of CD multiplied by DB to the square of AD is as ther atio of the parameter to the diameter, as proved in proposition twenty of book I of the Conic Sestions,[2] CD multiplied by DB becomes equal to the square of AD. The angle ABD is therefore equal to the angle CAD. And the angle DBA is equal to twice the angle ACD. For AB is equal to BC and therefore CAD is twice the angle ACD.

It results that the sum of the angles CAD and ACD is equal to three times the angle ACD. But the external angle ADV equals the sum of the internal angles DAC and ACD of the triangle. The angle ADV is thus three times the angle C, and the angle C is equal to one third of the angle ADV which is, by construction, the angle between the ordinates and the corresponding diameter of the hyperbola. Now, we had assumed this angle to be equal to the angle H. The angle C is thus one third of the angle H to be trisected. We have therefore found one third of the angle H. And this is what we wished to state.

End of ... the division of the angle into three equal parts by Abû Sahl al Kûhî, and gratitude is to God, the lord of the worlds, and may his blessing be upon his prophet Muḥammad and his noble descendents.

[1] Apollonios, *Conic Sections*, T.L. Heath, proposition 25, pp. 44-47.

[2] Apollonios, *ibid.*, see propositions 2 and 8, pp. 10, 19-20.